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A {1, 2}-ORDER THEORY FOR ELASTODYNAMIC ANALYSIS OF THICK ORTHOTROPIC SHELLS

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Abstract—A higher-order shell theory is developed for elastodynamic analysis of orthotropic shells. The theory accounts for all basic deformations including transverse shear and transverse normal strains and stresses. The theory is developed in orthogonal curvilinear coordinates in which the reference surface components of the displacement vector vary linearly through the thickness while the transverse displacement is parabolic. Transverse shear and transverse normal strains are formulated to satisfy physical traction conditions at the top and bottom shell surfaces, and are also made least-squares compatible with the corresponding strains that are derived directly from the strain-displacement relations of three-dimensional elasticity. In these variational statements of strain compatibility, transverse shear and transverse normal correction factors are introduced, and are determined from dynamic considerations in the manner originally proposed by Mindlin. Equations of motion and associated engineering (Poisson) boundary conditions are derived from a three-dimensional variational principle. An important feature of the present theory is the requirement of only simple C^0 and C^{-1} continuity for the shell kinematic variables. This aspect makes the theory particularly attractive for the development of efficient shell finite elements suitable for general purpose finite element analysis of thick shell structures. Analytical solutions for the free vibration of isotropic and orthotropic cylindrical shells are obtained for a wide range of thickness/radius and thickness/wavelength ratios and found to be in close agreement with the exact elasticity solutions.

NOTATION

A_1, A_2	surface metrics (Lamé parameters)
A_{ij}	shell membrane rigidities
B_{ij}	shell membrane-bending coupling rigidities
C_{ij}	elastic stiffness coefficients
D_{ij}	shell bending rigidities
G_{ij} $(i,j = 4, 5)$	shell transverse shear rigidities
C^0	the class of continuous functions possessing discontinuous derivatives at element interfaces
C^{-1}	the class of continuous functions that are discontinuous at element interfaces
I_i	inertial coefficients
L	axial half wavelength
$\mathbf{K}_{cyl}$	stiffness matrix of cylindrical shell
$\mathbf{M}_{cyl}$	mass matrix of cylindrical shell
a	cylindrical shell radius
$2h$	shell thickness
k_1, k_2	transverse shear correction factors
k_3, k_4	transverse normal correction factors
m, n	axial and circumferential wave numbers
u, v	midplane displacement along α_1 and α_2 directions
u_1, u_2, u_3	orthogonal displacement components
w, w_i $(i = 1, 2)$	components of the transverse displacement
x, θ, ζ	cylindrical coordinates
t	time variable
Ω_i	normalized frequency
δ	variational operator
ε_{ij}, κ_{ij}	strain and curvature components
θ_i $(i = 1, 2$, or $x, \theta)$	bending cross-sectional rotations
ξ	dimensionless thickness coordinate
σ_{ij}, τ_{ij}	stress components
α	$m\pi/L$
α_1, α_2, ζ	orthogonal curvilinear coordinates

ρ	material mass density
ω_i	circular frequency
$(\dot{\ })$	differentiation with respect to time
$()_{,i}$	partial differentiation with respect to x_i

1. INTRODUCTION

Thick-section composite laminates have found numerous structural applications in the areas of civil, aerospace and marine structural designs. Many of these structures can be classified as thick shells—the shells in which the thickness dimension is the same order of magnitude as the radii of curvature. Given these dimensional relations and, additionally, the consideration of the polymer-matrix graphite-fiber material constituents, such shells may exhibit significant transverse shear deformations effects. In addition, transverse normal deformations need also be accounted, especially in the areas of stress concentration and those regions where the span of loading is comparable to the thickness dimension. These transverse effects are known to be pronounced in high-frequency, short-wavelength dynamics.

Considering the fact that a vast majority of structural analysis today is performed with the use of general-purpose finite element codes, it becomes rather apparent that there is a need for a simple and accurate higher-order shell theory that is amenable to finite element approximations. Such a theory must take proper account of transverse shear and transverse normal deformations—the type of thickness deformations that can be significant in the response of thick shell structures to low-velocity impact and, in dynamics, high-frequency excitations.

The classical two-dimensional theories [e.g. see Love (1888); Naghdi (1956); Sanders (1959); Koiter (1960); Ambartsumyan (1964); Reissner and Stavsky (1961); Dong *et al.* (1962)] are governed by the Kirchhoff–Love assumption of negligible transverse shear and transverse normal deformations. They are known to provide adequate predictions of all response quantities in the elastostatic regime of thin shells; in elastodynamics, in addition to the thinness requirement, the practical range of applicability of the classical theory is restricted to low-frequency (long-wavelength) excitations. When applied to relatively thick shells and those subjected to high-frequency excitation, however, significant errors may result. This is generally true for homogeneous isotropic materials, and especially true for laminated composites which exhibit relatively weak stiffness properties in the direction transverse to the fiber orientation. In the latter circumstances, rather significant deformations may result in the transverse shear and normal directions, and the neglect of these effects in the approximate theory may no longer be appropriate.

The first-order shear deformation theories account for transverse shear deformation, yet they neglect the effect of transverse normal deformation [e.g. see Reissner (1944, 1945, 1985); Mindlin (1951); Dong and Tso (1972); Dong and Chun (1992); Reddy (1989); Khdeir *et al.* (1989)]. These two-dimensional theories have been used widely in the analysis of both homogeneous and composite shells because their applicability extends further into the moderately thick regime and higher-frequency dynamics. They also proved to be particularly useful in the realm of finite element approximations (Hughes, 1987). The major stimulus here is the requirement of lower-order continuity for the displacement variables. For these reasons, first-order shear deformation theories are employed almost exclusively in general-purpose commercial and research finite element codes.

Various higher-order theories have been proposed for the analysis of thick homogeneous and laminated shells. The majority of such theories provide higher-order displacement approximations to improve the inplane response and stress predictions, yet they neglect the effect of transverse normal deformations [see e.g. Reddy and Liu (1985); Whitney and Sun (1974); Di Sciuva (1987); Doxsee (1989)]. Other higher-order theories include the transverse normal effect but are penalized with a higher degree of complexity such as higher-order boundary conditions, a large number of equations of motion (equilibrium), and higher-order continuity requirements for finite element approximations [see e.g. Hildebrand *et al.* (1949); Naghdi (1957); Lo *et al.* (1977); Voyiadjis and Shi (1991)]. For these reasons, higher-order theories have been employed to a much lesser degree and

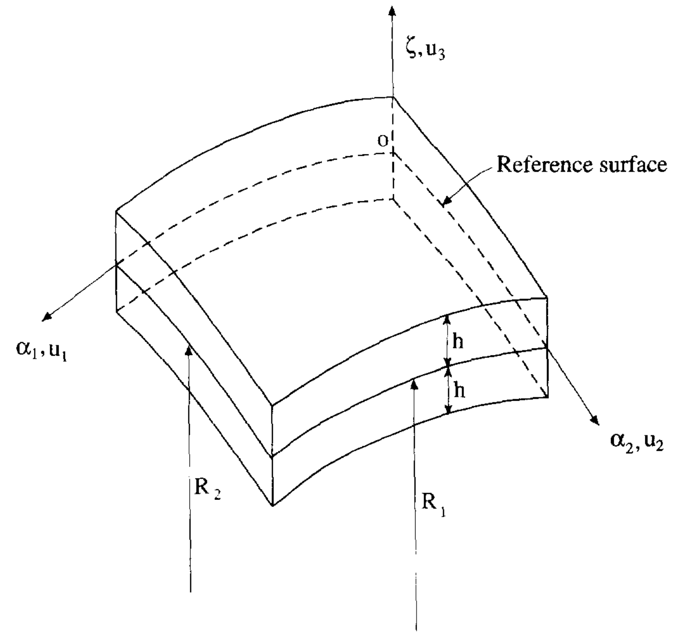

Fig. 1. Differential shell element.

have not found their way in general-purpose finite element programs. Only in certain relatively simple cases have three-dimensional elasticity solutions been successfully obtained [e.g. Nelson *et al.* (1971); Armenakas *et al.* (1969); Mirsky (1964, 1966)].

The main focus of this effort is to derive an accurate general shell theory which is particularly suited for finite element analysis and is applicable for the elastodynamics of thin and thick orthotropic shells. The approach is an extension of the {1, 2}-order plate theory of Tessler (1991, 1993) which accounts for transverse shear and transverse normal deformations, has a wide applicability range, possesses the simplicity of the first-order shear deformation theory, and is ideally suited for general-purpose finite element analysis. The present shell formulation may also serve as a foundation for a laminated composite theory following recent developments in plate theory [refer to Tessler and Saether (1991); Tessler *et al.* (1992, 1995)].

The proposed theory is evaluated via an analytic solution for the free vibration of isotropic and orthotropic cylindrical shells. Natural frequencies of vibration are determined for a wide range of geometric parameters, and the results are compared with the corresponding three-dimensional elasticity solutions.

2. FOUNDATION OF {1, 2}-ORDER SHELL THEORY

Let $(\alpha_1, \alpha_2, \zeta)$ denote an orthogonal curvilinear coordinate system of the shell of thickness $2h$, where α_1 and α_2 are the parametric, orthogonal lines of principal curvature of the shell reference midsurface, and $\zeta \in [-h, h]$ is the normal to the midsurface which is positioned at $\zeta = 0$. The principal radii of curvature of the reference surface are R_1 and R_2 (see Fig. 1). Also, let A_1 and A_2 denote the surface metrics of the shell element which are determined as

$$A_1^2 = \mathbf{r}_{,1} \cdot \mathbf{r}_{,1}, \quad A_2^2 = \mathbf{r}_{,2} \cdot \mathbf{r}_{,2}, \tag{1}$$

where $\mathbf{r}$ is the position vector of a point on the middle surface of the shell; hence, $A_1 = A_1(\alpha_1, \alpha_2)$ and $A_2 = A_2(\alpha_1, \alpha_2)$. To make the development meaningful for application to composite materials, the theory is carried out for an elastic, orthotropic material. It is further assumed that the deformations of the shell are small.

The displacement vector defined in the orthogonal curvilinear shell coordinate frame can be written as

$$\mathbf{U}(\alpha_1,\alpha_2,\zeta,t) = u_1(\alpha_1,\alpha_2,\zeta,t)\mathbf{i}_1+u_2(\alpha_1,\alpha_2,\zeta,t)\mathbf{i}_2+u_3(\alpha_1,\alpha_2,\zeta,t)\mathbf{n}, \tag{2}$$

where $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{n}$ are the unit vectors along α_1, α_2, and ζ, respectively.

Following the approach of a {1, 2}-order plate theory of Tessler (1991, 1993), the displacement components, which allow a three-dimensional deformation state including transverse shear and transverse normal deformations, are expanded in terms of the dimensionless thickness coordinate $\xi = \zeta/h \in [-1,\ 1]$ using seven kinematic variables, $\mathbf{u} = (u, v, w, \theta_1, \theta_2, w_1, w_2)$, as

$$\begin{aligned} u_1(\alpha_1,\alpha_2,\zeta,t) &= u(\alpha_1,\alpha_2,t)+h\xi\theta_1(\alpha_1,\alpha_2,t)\\ u_2(\alpha_1,\alpha_2,\zeta,t) &= v(\alpha_1,\alpha_2,t)+h\xi\theta_2(\alpha_1,\alpha_2,t)\\ u_3(\alpha_1,\alpha_2,\zeta,t) &= w(\alpha_1,\alpha_2,t)+\xi w_1(\alpha_1,\alpha_2,t)+(\xi^2+C)w_2(\alpha_1,\alpha_2,t), \end{aligned} \tag{3}$$

where t denotes time; C is a constant whose value is established by letting w be the weighted-average transverse deflection as in Reissner's first-order theory, i.e.

$$w = \int_{-h}^{h} u_3(1-\xi^2)\,\mathrm{d}\zeta. \tag{4}$$

The fulfilment of this condition requires that $C = -1/5$ (also, see Remark 1). As a result, the transverse displacement computed at the shell midsurface is defined by two dependent variables, i.e.

$$u_3(\alpha_1,\alpha_2,\zeta=0,t) = w(\alpha_1,\alpha_2,t)+Cw_2(\alpha_1,\alpha_2,t). \tag{5}$$

As in the first-order theory, the variables u, v, θ_1, and θ_2 can be interpreted as the weighted-average quantities,

$$(u,v) = \frac{1}{2h}\int_{-h}^{h}(u_1,u_2)\,\mathrm{d}\zeta, \quad (\theta_1,\theta_2) = \frac{3}{2h^3}\int_{-h}^{h}(u_1,u_2)\zeta\,\mathrm{d}\zeta. \tag{6}$$

The higher-order variables, w_1 and w_2, can be thought of as the normalized strain and curvature in the thickness direction, i.e.

$$w_1/h = u_{3,\zeta}(\zeta=0) = \varepsilon_n^0, \quad w_2/h^2 = \tfrac{1}{2}u_{3,\zeta\zeta} = \kappa_n^0. \tag{7}$$

The three-dimensional Hooke's law is assumed to govern the relationship between stresses and strains, which in matrix form may be written as

$$\begin{Bmatrix} \sigma_1\\ \sigma_2\\ \sigma_n\\ \tau_{2n}\\ \tau_{1n}\\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26}\\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36}\\ 0 & 0 & 0 & C_{44} & C_{45} & 0\\ 0 & 0 & 0 & C_{45} & C_{55} & 0\\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1\\ \varepsilon_2\\ \varepsilon_n\\ \gamma_{2n}\\ \gamma_{1n}\\ \gamma_{12} \end{Bmatrix}, \tag{8}$$

where the C_{ij} denote the elastic stiffness coefficients for an orthotropic material whose principal directions are not, in general, coincident with the shell coordinates α_1 and α_2, hence the presence of the shear coupling terms, C_{i6} ($i = 1, 2, 3$) and C_{45}.

The conventional computation of strains proceeds by introducing the assumed displacements into the strain-displacement relations of three-dimensional elasticity theory [e.g. see Kraus (1967)]. This gives rise to the following strain components.

The strains acting in the α_1 and α_2 coordinate directions,

$$\varepsilon_i = [\varepsilon_i^0 + h\xi\kappa_i^0 + h\xi\varepsilon_n^0/R_i + h^2(\xi^2 + C)\kappa_n^0/R_i]/L_i \quad (i = 1, 2)$$
$$\gamma_{12} = (\beta_1^0 + h\xi\beta_1')/L_1 + (\beta_2^0 + h\xi\beta_2')/L_2, \tag{9}$$

where $L_i = 1 + h\xi/R_i$ ($i = 1, 2$) and

$$\varepsilon_1^0 = u_{,1}/A_1 + vA_{1,2}/A_1A_2 + w/R_1$$
$$\varepsilon_2^0 = v_{,2}/A_2 + uA_{2,1}/A_1A_2 + w/R_2$$
$$\beta_1^0 = v_{,1}/A_1 - uA_{1,2}/A_1A_2$$
$$\beta_2^0 = u_{,2}/A_2 - vA_{2,1}/A_1A_2$$
$$\kappa_1^0 = \theta_{1,1}/A_1 + \theta_2A_{1,2}/A_1A_2$$
$$\kappa_2^0 = \theta_{2,2}/A_2 + \theta_1A_{2,1}/A_1A_2$$
$$\beta_1' = \theta_{2,1}/A_1 - \theta_1A_{1,2}/A_1A_2$$
$$\beta_2' = \theta_{1,2}/A_2 - \theta_2A_{2,1}/A_1A_2. \tag{10}$$

The transverse shear, γ_{in}($i = 1,2$), and transverse normal, ε_n, strains,

$$\gamma_{in} = \frac{1}{L_iA_i}[A_i\mu_i^0 + \xi w_{1,i} + (\xi^2 + C)w_{2,i}], \tag{11}$$

where

$$\mu_1^0 = -u/R_1 + \theta_1 + w_{,1}/A_1$$
$$\mu_2^0 = -v/R_2 + \theta_2 + w_{,2}/A_2$$

and

$$\varepsilon_n = \varepsilon_n^0 + 2\xi h\kappa_n^0. \tag{12}$$

Examining the distribution of the transverse shear strains in the thickness direction, as the shell approaches its thin limit $2h \to 0$, reveals the fulfillment of the limiting conditions $L_i \to 1$ and $w_{j,i} \ll \mu_i$. Under these constraints, the shear strains are practically uniform across the shell thickness, i.e. $\gamma_{in} \to \mu_i^0$. Hence, the conditions of zero shear stresses on the bounding shell faces cannot be achieved with these displacement assumptions. Further, associated with each shear strain, a correction factor needs to be specified to achieve agreement with the classical theory for the class of thin shell problems. This latter aspect is consistent with the shear correction notion in the first-order plate theory, Mindlin (1951). Also, as observed by Mindlin and Medick (1958) in the context of inplane plate vibrations, the thickness-motion response, which is characterized by the linear thickness variation of the transverse normal strain, ε_n, needs to be corrected. This is due to the sinusoidal character of the exact ε_n distribution which cannot be modeled with sufficient accuracy by the linear variation even for the lowest thickness-stretch mode. Here, two correction factors need to be specified—one associated with the constant and the other with the linear components in ε_n. The values of these corrections factors in the context of a {1,2}-order plate theory have been determined by Tessler *et al.* (1995), who employed the approaches of Mindlin (1951) and Mindlin and Medick (1958).

Rewriting eqns (11) and (12) with the use of transverse shear k_i ($i = 1,2$) and normal k_i ($i = 3,4$) correction factors results in the corrected transverse strains,

$$\gamma_{in} = \frac{k_i}{L_i A_i}[A_i \mu_i^0 + \xi w_{1,i} + (\xi^2 + C) w_{2,i}] \tag{13}$$

$$\varepsilon_n = k_3 \varepsilon_n^0 + k_4(2\xi h \kappa_n^0). \tag{14}$$

Once the strains are derived from the displacements in the manner described, the conventional displacement approach for elastodynamics is to employ Hamilton's principle. For this order of displacement approximation, Hamilton's principle gives rise to a set of seven second-order partial differential equations of motion and a set of variationally consistent boundary conditions. Such a theory is formally 14th-order, as are those derived in Hildebrand *et al.* (1949) and Naghdi (1957), and possesses higher-order boundary conditions associated with the w_1 and w_2 variables. Also, with the transverse shear strains defined by eqn (13), the associate shear stresses cannot fulfill zero shear traction conditions on the bounding shell surfaces. From the perspective of utilization of such a theory in a general-purpose finite element code, the large number of variables and the appearance of higher-order boundary conditions make it incompatible with the conventional, first-order theory framework, involving three displacement and two rotation variables. This explains the complete absence of higher-order theory shell elements in general-purpose finite element codes.

The aforementioned deficiencies, however, can be overcome by formulating the shell approximation in a multi-field manner, where in addition to the displacements, the transverse strains are also assumed independently. The manner in which this process is presently formulated will yield a simplified and accurate shell theory that also fulfills the needs of computational mechanics; that is a theory that is perfectly suited for finite element approximation (also, see Remarks 2). In what follows, this approximation approach is described.

We now propose an independent approximation of transverse shear strains of the following form

$$\gamma_{in}^*(\alpha_1, \alpha_2, \zeta, t) = \frac{1}{A_i L_i} \sum_{j=0}^{2} \gamma_{inj}(\alpha_1, \alpha_2, t)\xi^j \quad (i = 1, 2), \tag{15}$$

where γ_{inj} are yet unknown coefficients dependent on α_1, α_2 and t. Henceforth, the strains superscribed with the asterisk will represent the independently assumed strains to distinguish them from those derived directly from strain-displacement relations. The above assumptions allow the selection of the γ_{inj} strain coefficients in such a way as to satisfy exactly the zero shear equilibrium conditions on the top and bottom shell surfaces, i.e.

$$\tau_{in}(\alpha_1, \alpha_2, \pm h, t) = 0 \quad (i = 1, 2), \tag{16}$$

where Hooke's law, eqn (8), is used to obtain τ_{in} in terms of γ_{in}^*. The two homogeneous conditions for each shear stress determine two unknown coefficients for each shear strain in eqn (15). The remaining coefficient in each assumed strain is determined from the following variational statement in which γ_{in}^*, subject to the physical constraints, eqn (16), are made compatible across the shell thickness with the corresponding *corrected* shear strains, eqn (13); this compatibility is enforced in the least-squares sense as,

$$\text{minimize} \int_{-h}^{h} \left\{ \gamma_{in}^* - \frac{k_i}{L_i A_i} [A_i \mu_i^0 + \xi w_{1,i} + (\xi^2 + C) w_{2,i}] \right\}^2 A_i^2 L_i^2 \, d\zeta \quad (i = 1, 2), \tag{17}$$

where the integrals are minimized with respect to the unknown expansion coefficients. The resulting transverse shear strains take the simple form

$$\gamma_{in}^* = \frac{5k_i}{4} (1 - \xi^2) \mu_i^0 / L_i \quad (i = 1, 2). \tag{18}$$

Note that the γ_{in}^* strains are now defined exclusively in terms of the basic strain measures μ_i^0 and they possess no w_1 and w_2 contributions; this is contrasted with those shear strains which are conventionally derived from the strain-displacement relations (11) and (13). The analytic and computational benefits of the transverse strains just derived are further elucidated in Remark 3.

Similarly, an improved approximation for the transverse normal strain is introduced by assuming a cubic thickness distribution as

$$\varepsilon_n^*(\alpha_1, \alpha_2, \zeta, t) = \sum_{j=0}^{3} \varepsilon_{nj}(\alpha_1, \alpha_2, t) \xi^j. \tag{19}$$

Alternatively, a cubic σ_n could have been assumed—the two approaches being entirely equivalent for a homogeneous shell. The latter approach, however, has some advantages when laminated composite materials are considered (Tessler, 1993); whereas, the implementation of the former is somewhat more direct and simple.

To determine the unknown coefficients of the ε_n^* expansion, a homogeneous constraint condition is imposed on the transverse normal stress gradient as

$$\sigma_{n,\zeta}(\alpha_1, \alpha_2, \pm h, t) = 0. \tag{20}$$

This condition is an exact statement of transverse normal equilibrium for plates, i.e. when the initial curvatures are zero (Tessler, 1993), and it can only be regarded as an approximation for curved shells [e.g. refer to Sokolnikoff (1956) for the exact form of equilibrium equations in curvilinear coordinates]. It can further be argued that for shallow shells this approximation may still be adequate both as an average representation of the thickness stretch deformation as well as for computing the σ_n stress. For deep shells, however, the computation of σ_n directly from Hooke's law is not expected to be accurate. In these situations, σ_n may be obtained by integrating the exact equilibrium equations of three-dimensional elasticity theory—the procedure commonly used in recovering σ_n from classical and first-order theories. It is now worth pointing out that, as will be established in Results and Discussion, the theory produces accurate predictions of vibrational frequencies even for deep and very thick cylindrical shells. Since the accuracy of vibrational frequencies depends on how well the shell response is approximated in the average sense, it is reasonable to conclude that the enforcement of eqn (20) results in an adequate approximation of the thickness stretch deformation even for deep shells.

The remaining expansion coefficients in eqn (19) are determined by forcing ε_n^* to be least-squares compatible with the corrected ε_n strain of eqn (14),

$$\text{minimize} \int_{-h}^{h} \{\varepsilon_n^* - [k_3 \varepsilon_n^0 + k_4 (2\xi h \kappa_n^0)]\}^2 \, d\zeta. \tag{21}$$

This yields

$$\varepsilon_n^* = s_1\varepsilon_1^0 + s_2\varepsilon_2^0 + k_3 s_3\varepsilon_n^0 + s_4\beta_1^0 + s_5\beta_2^0 + s_6\kappa_1^0 + s_7\kappa_2^0 + k_4 s_8\kappa_n^0 + s_9\beta_1' + s_{10}\beta_2', \tag{22}$$

where the s_j ($j = 1, 2, \ldots, 10$) coefficients vary cubically with ξ and are also functions of the C_{ij} elastic constants (see Appendix A).

The complete kinematics of the shell can thus be expressed in terms of 12 reference-surface shell strain and curvature measures,

$$\boldsymbol{\varepsilon}_o^T = [\varepsilon_1^0, \varepsilon_2^0, \varepsilon_n^0, \beta_1^0, \beta_2^0], \quad \boldsymbol{\kappa}_o^T = [\kappa_2^0, \kappa_2^0, \kappa_n^0, \beta_1', \beta_2'], \quad \gamma_o^T = [\mu_2^0, \mu_1^0], \tag{23}$$

where $(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_n^0)$, (β_1^0, β_2^0) and (μ_1^0, μ_2^0) represent the normal, reference-surface shear and transverse shear strains; $(\kappa_1^0, \kappa_2^0, \kappa_n^0)$ and (β_1', β_2') denote the changes in the normal and twisting curvatures. The relationships of these quantities to the seven kinematic variables are given in eqns (7), (10) and (11).

Accounting for all strain and stress components and assuming no body forces, the equations of motion together with the natural boundary conditions are now derived by applying the three-dimensional variational principle,

$$\begin{aligned}
\delta \int_{t_0}^{t_1} \Bigg[\frac{1}{2} \int_{\alpha_1} \int_{\alpha_2} \int_{-h}^{h} & (\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_n\varepsilon_n^* + \tau_{12}\gamma_{12} + \tau_{1n}\gamma_{1n}^* + \tau_{2n}\gamma_{2n}^*) A_1 A_2 L_1 L_2 \,d\alpha_1\, d\alpha_2\, d\zeta \\
& - \int_{\alpha_1} \int_{\alpha_2} \int_{-h}^{h} \frac{\rho}{2} (\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2) A_1 A_2 L_1 L_2 \,d\alpha_1\, d\alpha_2\, d\zeta \\
& - \int_{\alpha_1} \int_{\alpha_2} (q_n^+ u_3^+ L_1^- L_2^+ - q_n^- u_3^- L_1^- L_2^-) A_1 A_2 \,d\alpha_1\, d\alpha_2 \\
& - \int_{\alpha_2} \int_{-h}^{h} (\bar{\sigma}_1 u_1 + \bar{\tau}_{12} u_2 + \bar{\tau}_{1n} u_3) A_2 L_2 \,d\alpha_2\, d\zeta \\
& - \int_{\alpha_1} \int_{-h}^{h} (\bar{\tau}_{21} u_1 + \bar{\sigma}_2 u_2 + \bar{\tau}_{2n} u_3) A_1 L_1 \,d\alpha_1\, d\zeta \Bigg] dt = 0,
\end{aligned} \tag{24}$$

where the quantities superscribed with a bar refer to the prescribed edge values, and the superscripts "+" and "−" respectively identify the appropriate quantities on the top and the bottom shell surfaces; q_n^+ and q_n^- are the normal tractions prescribed on the top and bottom bounding shell surfaces. Note that the first volume integral in eqn (24), which represents the strain energy in Hamilton's principle, has features of a mixed formulation—the mixed aspect is due to the inclusion of the independently assumed transverse strains. Unlike Reissner-type mixed formulations in which the assumed strains/stresses depend on the respective strain or stress functions, the assumed strains in the present formulation are functions of the displacement variables.

Integrating over the shell thickness results in the two-dimensional variational principle,

$$\delta \int_{t_0}^{t_1} \left\{ \frac{1}{2} \int_S [\mathbf{N}^T \boldsymbol{\varepsilon}_o + \mathbf{M}^T \boldsymbol{\kappa}_o + \mathbf{Q}^T \boldsymbol{\gamma}_o] \,d\alpha_1\, d\alpha_2 - K - W_e \right\} dt = 0, \tag{25}$$

where the vectors of stress resultants **N**, **M** and **Q** are given by

$$\begin{aligned}
\mathbf{N}^T &= [N_1, N_2, N_n, N_{12}, N_{21}] \\
\mathbf{M}^T &= [M_1, M_2, M_n, M_{12}, M_{21}] \\
\mathbf{Q}^T &= [Q_2, Q_1]
\end{aligned} \tag{26}$$

The stress resultants, expressed as integrated quantities of the stresses, are given in Appendix B.
The kinetic energy, K, and the external work, W_e, have the form

$$K = \frac{1}{2}\int_{\alpha_1}\int_{\alpha_2} [m_0(\dot{u}^2+\dot{v}^2+\dot{w}^2)+2hm_1(\dot{u}\dot{\theta}_1+\dot{v}\dot{\theta}_2)+h^2m_2(\dot{\theta}_1^2+\dot{\theta}_2^2)$$
$$+m_2\dot{w}_1^2+(m_4-2m_2/5+m_0/25)\dot{w}_2^2$$
$$+2m_1\dot{w}\dot{w}_1+2(m_2-m_0/5)\dot{w}\dot{w}_2$$
$$+2(m_3-m_1/5)\dot{w}_1\dot{w}_2]A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 \tag{27}$$

$$W_e = \int_{\alpha_1}\int_{\alpha_2}[q_1(w+4w_2/5)+q_2w_1]A_1A_2\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2+\int_{\alpha_1}(\bar{N}_1u+\bar{N}_{12}v+\bar{M}_1\theta_1+\bar{M}_{12}\theta_2$$
$$+\bar{Q}_1w+\bar{Q}_{11}w_1+\bar{Q}_{12}w_2)A_2\,\mathrm{d}\alpha_2$$
$$+\int_{\alpha_2}(\bar{N}_{21}u+\bar{N}_2v+\bar{M}_{21}\theta_1+\bar{M}_2\theta_2$$
$$+\bar{Q}_2w+\bar{Q}_{21}w_1+\bar{Q}_{22}w_2)A_1\,\mathrm{d}\alpha_1, \tag{28}$$

where

$$q_1 = q_n^+L_1^-L_2^+-q_n^-L_1^-L_2^-, \quad q_2 = q_n^+L_1^+L_2^++q_n^-L_1^-L_2^-, \tag{29}$$

The resultants of prescribed edge tractions ($\bar{N}_{ij}$, $\bar{M}_{ij}$, $\bar{Q}_{ij}$) and the m_i ($i = 0, 1, \ldots, 4$) inertial coefficients are defined respectively in Appendices C and E.

The shell constitutive relations are expressed as

$$\begin{Bmatrix}\mathbf{N}\\\mathbf{M}\\\mathbf{Q}\end{Bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B} & \mathrm{o}\\ \mathbf{B}^{\mathrm{T}} & \mathbf{D} & \mathrm{o}\\ \mathrm{o} & \mathrm{o} & \mathbf{G}\end{bmatrix}\begin{Bmatrix}\boldsymbol{\varepsilon}_{\mathrm{o}}\\ \boldsymbol{\kappa}_{\mathrm{o}}\\ \boldsymbol{\gamma}_{\mathrm{o}}\end{Bmatrix}, \tag{30}$$

where $\mathbf{A} = [A_{ij}]$, $\mathbf{B} = [B_{ij}]$, $\mathbf{D} = [D_{ij}]$ and $\mathbf{G} = [G_{ij}]$ are defined in Appendix D.

Performing appropriate variations and integration by parts yields the seven equations of motion,

$$(N_1A_2)_{,1}+(N_{21}A_1)_{,2}+N_{12}A_{1,2}-N_2A_{2,1}+A_1A_2Q_1/R_1 = A_1A_2(m_0\ddot{u}+m_1h\ddot{\theta}_1)$$
$$(N_{12}A_2)_{,1}+(N_2A_1)_{,2}+N_{21}A_{2,1}-N_1A_{1,2}+A_1A_2Q_2/R_2 = A_1A_2(m_0\ddot{v}+m_1h\ddot{\theta}_2)$$
$$(Q_1A_2)_{,1}+(Q_2A_1)_{,2}-(N_1/R_1+N_2/R_2)A_1A_2-q_nA_1A_2$$
$$= A_1A_2[m_0\ddot{w}+m_1\ddot{w}_1+(-m_0/5+m_2)\ddot{w}_2]$$
$$(M_1A_2)_{,1}+(M_{21}A_1)_{,2}+M_{12}A_{1,2}-M_2A_{2,1}-A_1A_2Q_1 = A_1A_2(m_1h\ddot{u}+m_2h^2\ddot{\theta}_1)$$
$$(M_{12}A_2)_{,1}+(M_2A_1)_{,2}+M_{21}A_{2,1}-M_1A_{1,2}-A_1A_2Q_2 = A_1A_2(m_1h\ddot{v}+m_2h^2\ddot{\theta}_2)$$
$$-N_n/h+q_2 = [m_1\ddot{w}+m_2\ddot{w}_1+(-m_1/5+m_3)\ddot{w}_2]$$
$$-M_n/h^2+4q_1/5 = [(-m_0/5+m_2)\ddot{w}+(-m_1/5+m_3)\ddot{w}_1+(m_0/25-2m_2/5+m_4)\ddot{w}_2] \tag{31}$$

and the Poisson-type boundary conditions that are consistent with the theory. The boundary conditions along the α_1 edge:

$$\begin{aligned} N_{21} &= \bar{N}_{21} \quad \text{or} \quad u = \bar{u} \\ N_2 &= \bar{N}_2 \quad \text{or} \quad v = \bar{v} \\ Q_2 &= \bar{Q}_2 \quad \text{or} \quad w = \bar{w} \\ M_{21} &= \bar{M}_{21} \quad \text{or} \quad \theta_1 = \bar{\theta}_1 \\ M_2 &= \bar{M}_2 \quad \text{or} \quad \theta_2 = \bar{\theta}_2. \end{aligned} \tag{32}$$

The boundary conditions along the α_2 edge

$$\begin{aligned} N_1 &= \bar{N}_1 \quad \text{or} \quad u = \bar{u} \\ N_{12} &= \bar{N}_{12} \quad \text{or} \quad v = \bar{v} \\ Q_1 &= \bar{Q}_1 \quad \text{or} \quad w = \bar{w} \\ M_{12} &= \bar{M}_{12} \quad \text{or} \quad \theta_1 = \bar{\theta}_1 \\ M_1 &= \bar{M}_1 \quad \text{or} \quad \theta_2 = \bar{\theta}_2 \end{aligned} \tag{33}$$

The variational principle also yields

$$\begin{aligned} \bar{Q}_{21} &= \bar{Q}_{22} = 0 \quad \text{along} \quad \alpha_1 \\ \bar{Q}_{11} &= \bar{Q}_{12} = 0 \quad \text{along} \quad \alpha_2 \end{aligned} \tag{33a}$$

To satisfy conditions (33a), the $\bar{\tau}_{1n}$ and $\bar{\tau}_{2n}$ shear tractions must be of the following form

$$\begin{aligned} \bar{\tau}_{1n} &= \bar{T}_{1n}(\alpha_2)(1-\xi^2)/L_2 \\ \bar{\tau}_{2n} &= \bar{T}_{2n}(\alpha_1)(1-\xi^2)/L_1 \end{aligned} \tag{34}$$

Remark 1. *Alternative procedure for determining the C coefficient*

The C coefficient appearing in the assumed displacement (3) can be alternatively determined from the least-squares statement (17) without the preceding enforcement of (4). Thus, the fulfillment of condition (17) with C treated as an unknown constant, results in the transverse shear strains which include terms associated with the gradients $w_{2,i}$ $(i = 1,2)$. In these expressions, if C is set to $-1/5$, the $w_{2,i}$ terms vanish identically, yielding the shear strains of eqn (18).

Remark 2. *Finite element approximation aspects*

The theory offers significant computational advantages as far as its finite element approximation is concerned. The basic issue is the interelement continuity requirement for kinematic variables associated with the theory. Here, the u, v, w, θ_1 and θ_2 kinematic variables, which are the same as in the first-order theory, need only be approximated with C^0-continuous shape functions; this is because their highest spatial derivatives appearing in the variational principle (25) do not exceed order one. Further, since the variational principle possesses no spatial gradients of the variables w_1 and w_2, their finite element approximations need only be C^{-1}-continuous, i.e. these fields can be discontinuous along finite element boundaries. With the latter assumptions, the w_1 and w_2 variables can be condensed out statically at the element level, thus giving rise to simple and computationally efficient elements. Such finite elements, based on the predecessor {1,2}-order plate theory, have been developed and successfully implemented in NASA's general-purpose finite element code COMET (Stewart, 1989), and used as a user-supplied element in ABAQUS (Hibbit *et al.*, 1992).

Remark 3. *Analytic simplicity characteristics*

The absence of the $w_{1,i}$ and $w_{2,i}$ gradients in the transverse shear strains and, subsequently, in the variational principle, results in the following simplifying features of the theory: (1) the resulting differential equations of motion are 10th order and not the usual 14th order for a seven-variable theory [e.g. Hildebrand *et al.* (1949), Naghdi (1957)];

and (2) the variationally derived boundary conditions are strictly of the Poisson type, without the usual higher-order boundary conditions that are necessarily present in a 14th-order theory.

3. TRANSVERSE CORRECTION FACTORS

The shell theory is completed upon determining the appropriate values for the correction factors k_i ($i = 1,4$) introduced in the formulation. The approach is the same as in the predecessor plate theory (Tessler *et al.*, 1995); it follows Mindlin's approach (Mindlin, 1951), by considering the free vibration of an infinite orthotropic plate. First, cut-off frequencies of the lowest thickness-shear modes computed from three-dimensional elasticity equations of motion and the present theory are matched, resulting in $k_1 = k_2 = \pi/\sqrt{10}$ (0.993). This value—which is nearly unity—implies an insignificant correction in shear and differs from that of Mindlin's first-order theory correction of $\pi/\sqrt{12}$ (0.907). Analogously, as in Mindlin and Medick (1958), cut-off frequencies for the lowest symmetric and anti-symmetric thickness-stretch modes obtained from three-dimensional elasticity theory and the present theory are matched, resulting in the values $k_3 = \pi/\sqrt{12}$ (0.907) and $k_4 = \pi/\sqrt{17/252}$ (0.816). These correction factors will subsequently be used in the analysis of general shells.

4. FREE VIBRATION OF CYLINDRICAL SHELLS

The present shell theory is evaluated by studying the free vibrations of isotropic and orthotropic homogeneous cylindrical shells. For a cylindrical shell of radius a, the equations of motion in terms of the displacement variables are obtained from eqn (31) by making use of the shell constitutive relations (30) and, subsequently, transforming the general curvilinear coordinates (α_1, α_2, ζ) to the circular cylindrical coordinates (x, θ, ζ), and by taking into account the appropriate geometric relations

$$A_1 = R_1 = \infty, \quad A_2 = R_2 = a, \quad \lim_{R_1 \to \infty} (R_1 \, \mathrm{d}\alpha_1) = \mathrm{d}x. \tag{35}$$

The seven equations of motion take the form

$$A_{11}u_{,xx} + A_{55}u_{,\theta\theta}/a^2 + (A_{12} + A_{54})v_{,x\theta}/a + B_{11}\theta_{\theta,xx} +$$
$$B_{55}\theta_{\theta,\theta\theta}/a^2 + (B_{12} + B_{54})\theta_{x,x\theta}/a + A_{12}w_{,x}/a + A_{13}w_{1,x}/h + B_{13}w_{2,x}/h^2 = m_0\ddot{u} + m_1 h\ddot{\theta}_\theta$$

$$(B_{54} + B_{12})u_{,x\theta}/a + B_{44}v_{,xx} + B_{22}v_{,\theta\theta}/a^2 + (D_{45} + D_{21})\theta_{\theta,x\theta}/a +$$
$$D_{44}\theta_{x,xx} + D_{22}\theta_{x,\theta\theta}/a^2 + (B_{22} - aG_{44})w_{,\theta}/a^2 + B_{32}w_{1,\theta}/ha +$$
$$D_{23}w_{2,\theta}/h^2 a + G_{44}(v - a\theta_x)/a = h(m_1\ddot{v} + m_2\ddot{\theta}_x)$$

$$G_{55}w_{,xx} + G_{44}w_{,\theta\theta}/a^2 - A_{21}u_{,x}/a - (A_{22} + G_{44})v_{,\theta}/a^2 -$$
$$(B_{21} - aG_{55})\theta_{\theta,x}/a - (B_{22} - aG_{44})\theta_{x,\theta}/a^2 - A_{22}w/a^2 - A_{23}w_1/ah$$
$$-B_{23}w_2/ah^2 = m_0\ddot{w} + m_1\ddot{w}_1 + (-m_0/5 + m_2)\ddot{w}_2]$$

$$(A_{45} + A_{21})u_{,x\theta}/a + A_{44}v_{,xx} + A_{22}v_{,\theta\theta}/a^2 + (B_{45} + B_{21})\theta_{\theta,x\theta}/a +$$
$$B_{44}\theta_{x,xx} + B_{22}\theta_{x,\theta\theta}a^2 + (A_{22} + G_{44})w_{,\theta}/a^2 + A_{23}w_{1,\theta}/ha +$$
$$B_{23}w_{2,\theta}/h^2 a - G_{44}(v - a\theta_x)/a^2 = m_0\ddot{v} + m_1 h\ddot{\theta}_x$$

$$B_{11}u_{,xx} + B_{55}u_{,\theta\theta}/a^2 + (B_{21} + B_{45})v_{,x\theta}/a + D_{11}\theta_{\theta,xx} +$$
$$D_{55}\theta_{\theta,\theta\theta}/a^2 + (D_{12} + D_{54})\theta_{x,x\theta}/a + (B_{21} - aG_{55})w_{,x}/a +$$

$$B_{31}w_{1,x}/h+D_{13}w_{2,x}/h^2-G_{55}\theta_\theta = h(m_1\ddot{u}+hm_2\ddot{\theta}_\theta)$$

$$-[A_{31}u_{,x}+A_{32}(v_{,\theta}+w)/a+A_{33}w_1/h+B_{31}\theta_{\theta,x}+B_{32}\theta_{x,\theta}/a+$$
$$B_{33}w_2/h^2]/h = m_1\ddot{w}+m_2\ddot{w}_1+(-m_1/5+m_3)\ddot{w}_2$$

$$-[B_{13}u_{,x}+B_{23}(v_{,\theta}+w)/a+B_{33}w_1/h+D_{31}\theta_{\theta,x}+D_{32}\theta_{x,\theta}/a+$$
$$D_{33}w_2/h^2]/h^2 = (-m_0/5+m_2)\ddot{w}+(-m_1/5+m_3)\ddot{w}_1+(m_0/25-2m_2/5+m_4)\ddot{w}_2. \quad (36)$$

For free vibration, the displacements are expanded in a modal infinite series as

$$u(x,\theta,t) = \sum_{m=n=1}^{\infty} U_{mn}\sin\alpha x\cos n\theta\, e^{i\omega_{mn}t}$$

$$v(x,\theta,t) = \sum_{m=n=1}^{\infty} V_{mn}\cos\alpha x\sin n\theta\, e^{i\omega_{mn}t}$$

$$w(x,\theta,t) = \sum_{m=n=1}^{\infty} W_{mn}\cos\alpha x\cos n\theta\, e^{i\omega_{mn}t}$$

$$\theta_x(x,\theta,t) = \sum_{m=n=1}^{\infty} \phi_{mn}\cos\alpha x\sin n\theta\, e^{i\omega_{mn}t}$$

$$\theta_\theta(x,\theta,t) = \sum_{m=n=1}^{\infty} \psi_{mn}\sin\alpha x\cos n\theta\, e^{i\omega_{mn}t}$$

$$w_1(x,\theta,t) = \sum_{m=n=1}^{\infty} W^1_{mn}\cos\alpha x\cos n\theta\, e^{i\omega_{mn}t}$$

$$w_2(x,\theta,t) = \sum_{m=n=1}^{\infty} W^2_{mn}\cos\alpha x\cos n\theta\, e^{i\omega_{mn}t} \quad (37)$$

For the special case $n = 0$, the displacement expansions are obtained by interchanging sin $n\theta$ and cos $n\theta$ in eqn (37). Substituting eqn (37) into eqn (36), results in the matrix eigenvalue equation

$$(\mathbf{K}_{\rm cyl}-\omega^2\mathbf{M}_{\rm cyl})\mathbf{\Delta} = 0, \quad (38)$$

with

$$\mathbf{\Delta}^{\rm T} = \{U_{mn}, V_{mn}, W_{mn}, \phi_{mn}, \psi_{mn}, W^1_{mn}, W^2_{mn}\},$$

where the coefficients of the stiffness $\mathbf{K}_{\rm cyl}$ and mass $\mathbf{M}_{\rm cyl}$ matrices are defined in Appendix E.

The equations of motion (36) and the resulting eigenvalue equations (38) (also refer to Appendix E) show that deformations through the thickness due to w_1 and w_2 couple with the stretching and bending shell deformations. This coupling is expected to be more pronounced in thick shells, and it is less significant in thin shells. Also, of particular interest is the fact that the first five equations of motion reduce to those of the first-order Mindlin-type theory once the coupling terms associated with the w_1 and w_2 variables (i.e. the K_{i6} and K_{i7} ($i = 1,2,\ldots,5$) stiffness and the M_{3j} ($j = 6,7$) mass terms, see Appendix E) are set to vanish. In addition, these "reduced" equations yield results consistent with the classical shell theory once the transverse shear rigidities G_{ij} ($i,j = 4,5$) are set to infinity.

Table 1. Normalized natural frequencies for homogeneous isotropic thin cylinders, $2h/a = 0.01$

$2h/L$	Ω_1 EXACT	Ω_1 HOT	Ω_2 EXACT	Ω_2 HOT	Ω_3 EXACT	Ω_3 HOT	Ω_4 EXACT	Ω_4 HOT
				$(m, n) = (1, 1)$				
0.01	0.0046	0.0046	0.01069	0.01069	1.0001	1.0002	1.8706	1.8708
0.10	0.0160	0.0159	0.1001	0.1001	1.0050	1.0050	1.8498	1.8726
0.20	0.0577	0.0577	0.2000	0.2000	1.0198	1.0198	1.8121	1.8779
0.40	0.1989	0.1986	0.4000	0.4000	1.0771	1.0771	1.7520	1.9020
				$(m, n) = (1, 3)$				
0.01	0.0026	0.0026	0.0142	0.0142	1.0001	1.0001	1.8709	1.8709
0.10	0.0160	0.0162	0.1005	0.1005	1.005	1.005	1.8496	1.8726
0.20	0.0579	0.0584	0.2002	0.2002	1.0199	1.0199	1.8120	1.8779
0.40	0.1990	0.2003	0.4001	0.4001	1.0771	1.0771	1.7520	1.9021

Table 2. Normalized natural frequencies for homogeneous isotropic thick cylinders, $2h/a = 0.3$

$2h/L$	Ω_1 EXACT	Ω_1 HOT	Ω_2 EXACT	Ω_2 HOT	Ω_3 EXACT	Ω_3 HOT	Ω_4 EXACT	Ω_4 HOT
				$(m, n) = (1, 1)$				
0.01	0.0012	0.0012	0.0972	0.0972	1.0083	1.0086	1.8583	1.8730
0.10	0.0616	0.0616	0.1648	0.1648	1.0183	1.0185	1.8423	1.8746
0.20	0.1266	0.1269	0.2375	0.2375	1.0383	1.0382	1.8100	1.8800
0.40	0.2440	0.2459	0.4142	0.4142	1.0970	1.0967	1.7551	1.9041
				$(m, n) = (1, 3)$				
0.01	0.0957	0.0957	0.2875	0.2875	1.0455	1.0458	1.7865	1.8858
0.10	0.1064	0.1065	0.3095	0.3095	1.0517	1.0520	1.7812	1.8877
0.20	0.1455	0.1457	0.3613	0.3613	1.0692	1.0697	1.7681	1.8938
0.40	0.2792	0.2801	0.5018	0.5018	1.1303	1.1308	1.7439	1.9208

5. RESULTS AND DISCUSSION

The free vibrations of isotropic cylinders are studied first. Tables 1 and 2 summarize selected natural frequencies for thin ($2h/a = 0.01$) and thick ($2h/a = 0.3$) isotropic cylinders corresponding to the mode numbers $(m, n) = (1, 1)$ and $(m, n) = (1, 3)$, respectively. In these tables, the frequencies are given in the range of thickness/length ratios of $0.01 \leqslant 2h/L \leqslant 0.40$. For each set of modal numbers (m, n) and the value of $2h/L$, four frequencies are furnished. The frequencies, $\Omega_i = \omega_i/\omega_{ref}$ $(i = 1,2,3,4)$ where $\omega_{ref} = \pi\sqrt{G/\rho}/2h$, correspond, in the ascending order, to a flexural mode, associated with large radial displacements; an axial shear mode, associated with large axial displacements; a thickness-shear mode, associated with motions in the axial directions, and a thickness-stretch mode exhibiting predominantly radial displacements. The present shell theory results are compared with those of three-dimensional elasticity theory (Armenakas *et al.*, 1969). The results for the first three frequencies (Ω_i, $i = 1,2,3$) are seen to be in excellent agreement with the exact solutions for both sets of modal numbers and the entire range of $2h/L$ examined. The Ω_4 frequency corresponding to the thickness-stretch mode is predicted accurately for thin and long shells; the shell theory tends to over estimate this frequency as the shell becomes thick and/or short. The largest error in the thickness-stretch frequency is about 10% corresponding to a short-thick cylinder ($2h/L = 0.40$ and $2h/a = 0.3$) and $n = 3$. Note that in the classical, first-order, and many higher-order theories, the thickness-stretch modes are suppressed entirely by virtue of the inextensibility assumption of the transverse normal fiber; hence, the thickness-stretch modes cannot be predicted by means of such formulations.

The second series of results concerns the axisymmetric vibrations of orthotropic cylinders, and these are summarized in Tables 3–5. Solutions are obtained for the aforementioned modes Ω_i $(i = 1,2,3,4)$ corresponding to $m = 1$ and $n = 0$ (i.e. axisymmetric motion). Thick cylinders with the $2h/a$ ratios of 0.25, 0.5 and 1.0 are analysed for a variety of thickness-to-wavelength ($2h/L$) ratios. The vibration frequencies are normalized as $\Omega_i = \omega_i/\omega_{ref}$ where $\omega_{ref} = 2h\sqrt{\rho/C_{55}}$. The shell theory predictions are compared with the exact solutions

Table 3. Normalized natural frequencies for homogeneous orthotropic topaz cylinders $(m, n) = (1, 0)$, $2h/a = 0.25$

$2h/L$	EXACT	HOT	HOT*	FSDT	CST
			Ω_1: Flexural mode		
0.00	0.0000	0.0000	0.0000	0.0000	0.0000
0.01	0.0878	0.0878	0.0878	0.0897	0.8972
0.1	0.8958	0.8954	0.8957	0.9429	0.9429
0.2	1.7777	1.7744	1.7774	1.8735	1.8735
0.3	2.6325	2.6208	2.6352	2.8073	2.8073
0.4	3.3981	3.3718	3.4296	3.7417	3.7417
0.5	3.8859	3.8883	4.0645	4.6763	4.6763
			Ω_2: Breathing mode		
0.00	0.3736	0.3736	0.3736	0.4060	0.4060
0.01	0.3738	0.3739	0.3739	0.4068	0.4068
0.1	0.3898	0.3902	0.3897	0.4105	0.4121
0.2	0.6511	0.6471	0.6452	0.6690	0.7266
0.3	1.1681	1.1187	1.1163	1.1399	1.3733
0.4	1.7523	1.6786	1.6771	1.6951	2.2078
0.5	2.8023	2.2731	2.2740	2.2824	3.1426
			Ω_3: Thickness-shear mode		
0.00	3.1490	3.1498	3.1705	3.1498	—
0.01	3.1506	3.1515	3.1722	3.1517	—
0.1	3.3088	3.3110	3.3327	3.3319	—
0.2	3.7158	3.7221	3.7480	3.7975	—
0.3	4.2538	4.2663	4.3025	4.4232	—
0.4	4.8466	4.8721	4.9245	5.1364	—
0.5	5.4468	5.4847	5.5777	5.9023	—
			Ω_4: Thickness-stretch mode		
0.00	4.5864	4.5726	5.0414	—	—
0.01	4.5853	4.5726	5.0415	—	—
0.1	4.5383	4.5810	5.0491	—	—
0.2	4.4461	4.6109	5.0752	—	—
0.3	4.4027	4.6808	5.1325	—	—
0.4	4.5244	4.8465	5.2575	—	—
0.5	4.9799	5.2559	5.5431	—	—

specifically formulated for axisymmetric vibrations of generally orthotropic cylindrical shells (Mirsky, 1964). The exact solution is based on a Frobenius power-series solution to the governing equations of three-dimensional elasticity; it demonstrates rapid convergence for the very thick case $2h/a = 1$. Convergence difficulties are, however, encountered for the low $2h/a$ and high $2h/L$ ratios for which an asymptotic solution is developed, Mirsky (1966). Both formulations presented by Mirsky (1964, 1966) have been implemented in this effort in order to examine a wider range of shell geometric parameters and material properties than the ones reported in the original references. The material properties of topaz are used herein to compare with results contained in the original references. The material moduli are given by:

$$C_{11} = 3005 \quad C_{12} = 900 \quad C_{13} = 864$$
$$C_{22} = 3561 \quad C_{23} = 1284 \quad C_{33} = 2871$$
$$C_{44} = 1100 \quad C_{55} = 1357 \quad C_{66} = 1330.$$

In Tables 3–5, comparisons are made with solutions based on the three-dimensional elasticity theory (denoted as EXACT), the present higher-order theory with and without the application of transverse correction factors (designated HOT and HOT*, respectively), the first-order shear deformable theory (FSDT), and classical shell theory (CST). It is again necessary to point out that the predictive capabilities of FSDT and CST are known to be adequate in the range of low frequencies and small $2h/a$ ratios. The range of thickness-to-radius ratio that is examined, $2h/a \geqslant 0.25$, falls into the category of thick shells for which these two theories are generally not suitable. The comparison with FSDT and CST, however, is useful in order to assertain quantitatively both the kind of error that is incurred

Table 4. Normalized natural frequencies for homogeneous orthotropic topaz cylinders $(m, n) = (1, 0)$, $2h/a = 0.5$

$2h/L$	EXACT	HOT	HOT*	FSDT	CST
		Ω_1 : Flexural mode			
0.00	0.0000	0.0000	0.0000	0.0000	0.0000
0.01	0.0879	0.0879	0.0879	0.0899	0.0899
0.1	0.9254	0.9254	0.9254	1.0000	1.0001
0.2	1.7851	1.7828	1.7828	1.8857	1.8857
0.3	2.6394	2.6300	2.6435	2.8144	2.8144
0.4	3.4050	3.3865	3.4418	3.7468	3.7468
0.5	3.9106	3.9120	4.0849	4.6804	4.6804
		Ω_2 : Breathing mode			
0.00	0.7635	0.7638	0.7638	0.8187	0.8187
0.01	0.7635	0.7637	0.7638	0.8189	0.8189
0.1	0.7235	0.7241	0.7236	0.7349	0.7349
0.2	0.8842	0.8873	0.8848	0.9170	0.9461
0.3	1.2580	1.2617	1.2582	1.2884	1.4693
0.4	1.7728	1.7714	1.7686	1.7905	2.2435
0.5	2.3574	2.3395	2.3392	2.3492	3.1495
		Ω_3 : Thickness-shear mode			
0.00	3.0893†	3.1748	3.1957	3.1748	—
0.01	3.1731†	3.1766	3.1975	3.1768	—
0.1	3.3335	3.3383	3.3603	3.3590	—
0.2	3.7436	3.7525	3.7791	3.8269	—
0.3	4.2823†	4.2985	4.3357	4.4528	—
0.4	4.8768†	4.9036	4.9590	5.1648	—
0.5	5.4770†	5.5196	5.6123	5.9291	—
		Ω_4 : Thickness-stretch mode			
0.00	4.6406†	4.5850	5.0538	—	—
0.01	4.6400†	4.5851	5.0539	—	—
0.1	4.5915†	4.5932	5.0612	—	—
0.2	4.4956†	4.6219	5.0864	—	—
0.3	4.4873†	4.6890	5.1415	—	—
0.4	4.7935†	4.8494	5.2618	—	—
0.5	4.9839†	5.2463	5.5395	—	—

†Denotes an asymptotic three-dimensional elasticity solution.

with the use of such theories and the benefits of utilizing the present shell theory for the analysis of thick shells. The results shown in the tables demonstrate highly accurate frequency predictions obtained with the present higher-order theory in the range of long and short cylinders examined. The results are also consistent and demonstrate that, with increasing $2h/a$ and $2h/L$ ratios, CST shows the greatest inaccuracy and, as noted previously, it does not permit solutions for the thickness-shear and thickness-stretch modes. The FSDT, in which thickness-stretch modes are intrinsically suppressed and which makes use of Mindlin's shear correction factors, $k_1 = k_2 = \pi/\sqrt{12}$, demonstrates a stronger performance in maintaining accuracy with increasing thickness/radius and thickness/wavelength ratios. Without the inclusion of transverse correction factors, the HOT* results are less accurate than the HOT formulation; it is noteworthy, however, that even without the shear correction factor, HOT* provides more accurate predictions for Ω_1 and Ω_3 than FSDT. The present theory, with the application of transverse correction factors, demonstrates exceptional performance in high thickness cylindrical shell geometries when compared with the exact solution.

6. CONCLUDING REMARKS

A new higher-order shell theory, which has analytic and computational advantages over other theories of the same order of approximation, was developed for homogeneous

Table 5. Normalized natural frequencies for homogeneous orthotropic topaz cylinders $(m, n) = (1, 0)$, $2h/a = 1.0$

$2h/L$	EXACT	HOT	HOT*	FSDT	CST
			Ω_1 : Flexural Mode		
0.00	0.0000	0.0000	0.0000	0.0000	0.0000
0.01	0.0879	0.0879	0.0879	0.0935	0.0902
0.1	0.8695	0.8692	0.8696	0.8886	0.8886
0.2	1.5040	1.5097	1.5091	1.5104	1.5109
0.3	1.7548	1.7663	1.7614	1.7771	1.8345
0.4	2.1252	2.1367	2.1317	2.1489	2.4078
0.5	2.6005	2.6059	2.6048	2.6171	3.1944
			Ω_2 : Breathing mode		
0.00	1.6532	1.6519	1.6460	1.6979	1.6980
0.01	1.6528	1.6666	1.6662	1.6975	1.6977
0.1	1.6338	1.6471	1.6452	1.6862	1.6930
0.2	1.8937	1.8995	1.8970	2.0007	2.0019
0.3	2.6981	2.6924	2.6991	2.8530	2.8533
0.4	3.4984	3.4749	3.5121	3.7704	3.7705
0.5	4.1118	4.0659	4.2059	4.6976	4.6977
			Ω_3 : Thickness-shear mode		
0.00	3.2710	3.2812	3.3029	3.2813	—
0.01	3.2730	3.2831	3.3048	3.2833	—
0.1	3.4446	3.4581	3.4813	3.4771	—
0.2	3.8679	3.8892	3.9179	3.9571	—
0.3	4.3962	4.4384	4.4800	4.5837	—
0.4	4.7137	4.9077	5.0892	5.2902	—
0.5	5.0576	5.2529	5.5615	6.0467	—
			Ω_4 : Thickness-stretch mode		
0.00	4.9547	4.7131	5.1901	—	—
0.01	4.9548	4.7131	5.1902	—	—
0.1	4.8939	4.7203	5.1970	—	—
0.2	4.7774	4.7456	5.2204	—	—
0.3	4.7170	4.8049	5.2711	—	—
0.4	5.0436	5.0693	5.3873	—	—
0.5	5.6297	5.6718	5.7986	—	—

orthotropic shells on the basis of assumed displacements and transverse strains. The independently assumed transverse shear and normal strains were derived in terms of displacement variables in two basic stages: (1) by enforcing physical transverse stress conditions to be exactly satisfied on the bounding shell surfaces; and (2) by making these transverse strains to be least-squares compatible across the shell thickness with the corresponding strains derived from the strain-displacement relations. The application of a three-dimensional displacement-based variational principle resulted in a 10th-order shell theory with associated five edge boundary conditions of the Poisson type. The theory is formulated in terms of an orthogonal curvilinear coordinate system and thus permits the analysis of various types of shells including, but not limited to, cylindrical, spherical and conical.

The analytic predictions of the shell theory for the natural frequencies of free vibration of isotropic and orthotropic cylindrical shells were found to be in close agreement with the three-dimensional elasticity solutions. In addition to the modes of deformation available in the first-order shear-deformable theory, the present theory incorporates two lowest thickness-stretch modes. The ability to model these thickness-stretch modes may be particularly important for laminated polymer-matrix composite shells, where the excitation of thickness-stretch modes is often associated with delamination initiation and failure. Naturally, the proposed theory provides a basic foundation for the development of a laminated shell theory.

The proposed theory may be found to be particularly useful for application to finite element analysis. The key appealing features are the low-order continuity requirements for the kinematic variables of the theory and the standard engineering boundary conditions. These characteristics permit formulations of simple and effective shell elements that are fully compatible with standard finite element software. The utility of such finite elements has recently been demonstrated with its predecessor plate theory, on which basis an effective faceted shell element has been developed and used in general-purpose finite element codes.

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APPENDIX A: DEFINITIONS OF s_i

The coefficients s_i ($i = 1, 2, \ldots, 10$) in eqn (22) are defined as follows:

$$s_i = \bar{P}_2 a_i + \bar{P}_1 b_i \quad (i \neq 3, 8)$$
$$s_3 = \bar{P}_2 a_3 + \bar{P}_1 b_3 + 1$$
$$s_8 = \bar{P}_2 a_8 + \bar{P}_1 b_8 + \bar{P}_3,$$

where

$$\bar{P}_1 = P_2(\xi)/h$$
$$\bar{P}_2 = [P_1(\xi) + 14P_3(\xi)]/85h$$
$$\bar{P}_3 = [168P_1(\xi) - 28P_3(\xi)]h/85$$

and $P_i(\xi)$ ($i = 1, 2, 3$) are the Legendre polynomials

$$P_1(\xi) = \xi$$
$$P_2(\xi) = (3\xi^2 - 1)/2$$
$$P_3(\xi) = \xi(5\xi^2 - 3)/2.$$

The a_i coefficients are defined as

$$a_i = h^2/R_i(C_{i3}/C_{33})K_i/2 \quad (i = 1, 2)$$
$$a_3 = h^3/R_1(\bar{K}_1/R_1 - K_3/h)(C_{13}/C_{33})/2 + h^3/R_2(\bar{K}_2/R_2 - K_4/h)(C_{23}/C_{33})/2$$
$$a_{3+i} = (C_{36}/C_{i3})/a_i$$
$$a_{5+i} = h^2/R_i(h\bar{K}_i - R_i K_{2+i})(C_{i3}/C_{33})/2$$
$$a_8 = h^4/R_1(4K_1/5R_1 - 2\bar{K}_3/h)(C_{13}/C_{33})/2 + h^4/R_2(4K_2/5R_2 - 2\bar{K}_4/h)(C_{23}/C_{33})/2$$
$$a_{8+i} = (C_{36}/C_{i3})/a_{5+i},$$

where the K_i and $\bar{K}_i$ are defined as follows:

$$K_i = 1/(L_i^+)^2 + 1/(L_i^-)^2$$
$$\bar{K}_i = 1/(L_i^+)^2 - 1/(L_i^-)^2$$
$$K_{2+i} = 1/L_i^+ + 1/L_i^-$$
$$\bar{K}_{2+i} = 1/L_i^+ - 1/L_i^-.$$

To obtain b_i, replace K_i with $\bar{K}_i$ and $\bar{K}_i$ with K_i in a_i ($i = 1, 2, 3, \ldots, 10$).

APPENDIX B: SHELL STRESS RESULTANTS

$$N_1 = h\int_{-1}^{1} (\sigma_1 + \sigma_n s_1 L_1)L_2 \, d\xi$$

$$N_2 = h\int_{-1}^{1} (\sigma_2 + \sigma_n s_2 L_2)L_1 \, d\xi$$

$$N_n = h\int_{-1}^{1} [(\sigma_1/R_1 + \sigma_2 L_1/L_2 R_2)h\xi + \sigma_n s_3 L_1]L_2 \, d\xi$$

$$N_{12} = h\int_{-1}^{1}(\tau_{12}+\sigma_n s_4 L_1)L_2\,\mathrm{d}\xi$$

$$N_{21} = h\int_{-1}^{1}(\tau_{21}+\sigma_n s_5 L_2)L_1\,\mathrm{d}\xi$$

$$M_1 = h\int_{-1}^{1}(\sigma_1 h\xi+\sigma_n s_6 L_1)L_2\,\mathrm{d}\xi$$

$$M_2 = h\int_{-1}^{1}(\sigma_2 h\xi+\sigma_n s_7 L_2)L_1\,\mathrm{d}\xi$$

$$M_n = h\int_{-1}^{1}[(\sigma_1/R_1+\sigma_2 L_1/L_2 R_2)h^2(\xi^2-1/5)+\sigma_n s_8 L_1]L_2\,\mathrm{d}\xi$$

$$M_{12} = h\int_{-1}^{1}(\tau_{12} h\xi+\sigma_n s_9 L_1)L_2\,\mathrm{d}\xi$$

$$M_{21} = h\int_{-1}^{1}(\tau_{21} h\xi+\sigma_n s_{10} L_2)L_1\,\mathrm{d}\xi$$

$$Q_1 = 5h/4\int_{-1}^{1}\tau_{1n}(1-\xi^2)L_2\,\mathrm{d}\xi$$

$$Q_2 = 5h/4\int_{-1}^{1}\tau_{2n}(1-\xi^2)L_1\,\mathrm{d}\xi$$

APPENDIX C: RESULTANTS OF PRESCRIBED EDGE TRACTIONS

$$\bar{N}_1 = h\int_{-1}^{1}\bar{\sigma}_1 L_2\,\mathrm{d}\xi$$

$$\bar{N}_2 = h\int_{-1}^{1}\bar{\sigma}_2 L_1\,\mathrm{d}\xi$$

$$\bar{N}_{12} = h\int_{-1}^{1}\bar{\tau}_{12} L_2\,\mathrm{d}\xi$$

$$\bar{N}_{21} = h\int_{-1}^{1}\bar{\tau}_{21} L_1\,\mathrm{d}\xi$$

$$\bar{M}_1 = h^2\int_{-1}^{1}\bar{\sigma}_1 L_2\xi\,\mathrm{d}\xi$$

$$\bar{M}_2 = h^2\int_{-1}^{1}\bar{\sigma}_2 L_1\xi\,\mathrm{d}\xi$$

$$\bar{M}_{12} = h^2\int_{-1}^{1}\bar{\tau}_{12} L_2\xi\,\mathrm{d}\xi$$

$$\bar{M}_{21} = h^2\int_{-1}^{1}\bar{\tau}_{21} L_1\xi\,\mathrm{d}\xi$$

$$\bar{Q}_1 = h\int_{-1}^{1}\bar{\tau}_{1n} L_2\,\mathrm{d}\xi$$

$$\bar{Q}_2 = h\int_{-1}^{1}\bar{\tau}_{2n} L_1\,\mathrm{d}\xi$$

$$\bar{Q}_{11} = h\int_{-1}^{1}\bar{\tau}_{1n} L_2\xi\,\mathrm{d}\xi = 0$$

$$\bar{Q}_{12} = h\int_{-1}^{1} \bar{\tau}_{1n}(\xi^2-1/5)L_2\,\mathrm{d}\xi = 0$$

$$\bar{Q}_{21} = h\int_{-1}^{1} \bar{\tau}_{2n}L_1\xi\,\mathrm{d}\xi = 0$$

$$\bar{Q}_{22} = h\int_{-1}^{1} \bar{\tau}_{2n}(\xi^2-1/5)L_1\,\mathrm{d}\xi = 0$$

The last four terms vanish identically, according to the natural boundary conditions of eqn (33a), which necessitate the prescribed shear tractions of the form shown by eqn (34).

APPENDIX D: SHELL CONSTITUTIVE MATRICES

The components of the shell theory constitutive matrices, eqn (30), are given as

$$A_{11} = h\int_{-1}^{1} [C_{11}/L_1 + C_{13}s_1 + (C_{13}/L_1 + C_{33}s_1)s_1L_1]L_2\,\mathrm{d}\xi$$

$$A_{12} = h\int_{-1}^{1} [C_{12}/L_1 + C_{13}s_2 + (C_{23}/L_1 + C_{33}s_2)s_1L_1]L_2\,\mathrm{d}\xi$$

$$A_{13} = k_3h\int_{-1}^{1} [C_{11}h\xi/L_1R_1 + C_{12}h\xi/L_2R_2 + C_{13}s_3 + (C_{13}h\xi/L_1R_1 + C_{23}h\xi/L_2R_2 + C_{33}s_3)s_1L_1]L_2\,\mathrm{d}\xi$$

$$A_{14} = h\int_{-1}^{1} [C_{13}s_4 + C_{16}/L_1 + (C_{33}s_4 + C_{36}/L_1)s_1L_1]L_2\,\mathrm{d}\xi$$

$$A_{15} = h\int_{-1}^{1} [C_{13}s_5 + C_{16}/L_2 + (C_{33}s_5 + C_{36}/L_2)s_1L_1]L_2\,\mathrm{d}\xi$$

$$A_{22} = h\int_{-1}^{1} [C_{22}/L_2 + C_{23}s_2 + (C_{23}/L_2 + C_{33}s_2)s_2L_2]L_1\,\mathrm{d}\xi$$

$$A_{23} = k_3h\int_{-1}^{1} [C_{12}h\xi/L_1R_1 + C_{22}h\xi/L_2R_2 + C_{23}s_3 + (C_{13}h\xi/L_1R_1 + C_{23}h\xi/L_2R_2 + C_{33}s_3)s_2L_2]L_1\,\mathrm{d}\xi$$

$$A_{24} = h\int_{-1}^{1} [C_{23}s_4 + C_{26}/L_1 + (C_{33}s_4 + C_{36}/L_1)s_2L_2]L_1\,\mathrm{d}\xi$$

$$A_{25} = h\int_{-1}^{1} [C_{23}s_5 + C_{26}/L_2 + (C_{33}s_5 + C_{36}/L_2)s_2L_2]L_1\,\mathrm{d}\xi$$

$$A_{33} = k_3^2h\int_{-1}^{1} [(C_{11}h\xi/L_1R_1 + C_{12}h\xi/L_2R_2 + C_{13}s_3)h\xi/R_1 + (C_{12}h\xi/L_1R_1 + C_{22}h\xi/L_2R_2 + C_{23}s_3)h\xi L_1/R_1L_2$$
$$+ (C_{13}h\xi/L_1R_1 + C_{23}h\xi/L_2R_2 + C_{33}s_3)s_3L_1]L_2\,\mathrm{d}\xi$$

$$A_{34} = k_3h\int_{-1}^{1} [(C_{13}s_4 + C_{16}/L_1)h\xi/R_1 + (C_{23}s_4 + C_{26}/L_1)h\xi L_1/L_2R_2 + (C_{33}s_4 + C_{36}/L_1)s_3L_1)]L_2\,\mathrm{d}\xi$$

$$A_{35} = k_3h\int_{-1}^{1} [(C_{13}s_5 + C_{16}/L_2)h\xi/R_1 + (C_{23}s_5 + C_{26}/L_2)h\xi L_1/L_2R_2 + (C_{33}s_5 + C_{36}/L_2)s_3L_1)]L_2\,\mathrm{d}\xi$$

$$A_{44} = h\int_{-1}^{1} [C_{66}/L_1 + (C_{33}s_4 + 2C_{36}/L_1)s_4L_1]L_2\,\mathrm{d}\xi$$

$$A_{45} = h\int_{-1}^{1} [C_{66}/L_2 + C_{36}s_5 + (C_{33}s_5 + C_{36}/L_2)s_4L_1]L_2\,\mathrm{d}\xi$$

$$A_{55} = h\int_{-1}^{1} [C_{66}/L_2 + (C_{33}s_5 + 2C_{36}/L_2)s_5L_2]L_1\,\mathrm{d}\xi$$

$$B_{11} = h\int_{-1}^{1} [C_{11}h\xi/L_1 + C_{13}s_6 + (C_{13}h\xi/L_1 + C_{33}s_6)s_1L_1]L_2\,\mathrm{d}\xi$$

$$B_{12} = h\int_{-1}^{1} [C_{12}h\xi/L_2 + C_{13}s_7 + (C_{23}h\xi/L_2 + C_{33}s_7)s_1 L_1]L_2 \, d\xi$$

$$B_{13} = k_4 h\int_{-1}^{1} \{C_{11}h^2(\xi^2 - 1/5)/L_1 R_1 + C_{12}h^2(\xi^2 - 1/5)/L_2 R_2 + C_{13}s_8 + [C_{13}h^2(\xi^2 - 1/5)/L_1 R_1 + C_{23}h^2(\xi^2 - 1/5)/L_2 R_2 + C_{33}s_8]s_1 L_1\} L_2 \, d\xi$$

$$B_{14} = h\int_{-1}^{1} [C_{13}s_9 + C_{16}h\xi/L_1 + (C_{33}s_9 + C_{36}h\xi/L_1)s_1 L_1]L_2 \, d\xi$$

$$B_{15} = h\int_{-1}^{1} [C_{13}s_{10} + C_{16}h\xi/L_2 + (C_{33}s_{10} + C_{36}h\xi/L_2)s_1 L_1]L_2 \, d\xi$$

$$B_{21} = h\int_{-1}^{1} [C_{12}h\xi/L_1 + C_{23}s_6 + (C_{13}h\xi/L_1 + C_{33}s_6)s_2 L_2]L_1 \, d\xi$$

$$B_{22} = h\int_{-1}^{1} [C_{22}h\xi/L_2 + C_{23}s_7 + (C_{23}h\xi/L_2 + C_{33}s_7)s_2 L_2]L_1 \, d\xi$$

$$B_{23} = k_4 h\int_{-1}^{1} \{C_{12}h^2(\xi^2 - 1/5)/L_1 R_1 + C_{22}h^2(\xi^2 - 1/5)/L_2 R_2 + C_{23}s_8 + [C_{13}h^2(\xi^2 - 1/5)/L_1 R_1 + C_{23}h^2(\xi^2 - 1/5)/L_2 R_2 + C_{33}s_8]s_2 L_2\} L_1 \, d\xi$$

$$B_{24} = h\int_{-1}^{1} [C_{23}s_9 + C_{26}h\xi/L_1 + (C_{33}s_9 + C_{36}h\xi/L_1)s_2 L_2]L_1 \, d\xi$$

$$B_{25} = h\int_{-1}^{1} [C_{23}s_{10} + C_{26}h\xi/L_2 + (C_{33}s_{10} + C_{36}h\xi/L_2)s_2 L_2]L_1 \, d\xi$$

$$B_{31} = k_3 h\int_{-1}^{1} [(C_{11}h\xi/L_1 + C_{13}s_6)h\xi/R_1 + (C_{12}h\xi/L_1 + C_{23}s_6)h\xi L_1/L_2 R_2 + (C_{13}h\xi/L_1 + C_{33}s_6)s_3 L_1]L_2 \, d\xi$$

$$B_{32} = k_3 h\int_{-1}^{1} [(C_{12}h\xi/L_2 + C_{13}s_7)h\xi/R_1 + (C_{22}h\xi/L_2 + C_{23}s_7)h\xi L_1/L_2 R_2 + (C_{23}h\xi/L_2 + C_{33}s_7)s_3 L_1]L_2 \, d\xi$$

$$B_{33} = k_3 k_4 h\int_{-1}^{1} \{[C_{11}h^2(\xi^2 - 1/5)/L_1 R_1 + C_{12}h^2(\xi^2 - 1/5)/L_2 R_2 + C_{13}s_8]h\xi/R_1 + [C_{12}h^2(\xi^2 - 1/5)/L_1 R_1 + C_{22}h^2(\xi^2 - 1/5)/L_2 R_2 + C_{23}s_8]h\xi L_1/L_2 R_2 + [C_{13}h^2(\xi^2 - 1/5)/L_1 R_1 + C_{23}h^2(\xi^2 - 1/5)/L_2 R_2 + C_{33}s_8]s_3 L_1\} L_2 \, d\xi$$

$$B_{34} = k_3 h\int_{-1}^{1} [(C_{13}s_9 + C_{16}h\xi/L_1)h\xi/R_1 + (C_{23}s_9 + C_{26}h\xi/L_1)h\xi L_1/L_2 R_2 + (C_{33}s_9 + C_{36}h\xi/L_1)s_3 L_1]L_2 \, d\xi$$

$$B_{35} = k_3 h\int_{-1}^{1} [(C_{13}s_{10} + C_{16}h\xi/L_2)h\xi/R_1 + (C_{23}s_{10} + C_{26}h\xi/L_2)h\xi L_1/L_2 R_2 + (C_{33}s_{10} + C_{36}h\xi/L_2)s_3 L_1]L_2 \, d\xi$$

$$B_{41} = h\int_{-1}^{1} [(C_{13}h\xi/L_1 + C_{33}s_6)s_4 L_1 + C_{16}h\xi/L_1 + C_{36}s_6]L_2 \, d\xi$$

$$B_{42} = h\int_{-1}^{1} [(C_{23}h\xi/L_2 + C_{33}s_7)s_4 L_1 + C_{26}h\xi/L_2 + C_{36}s_7]L_2 \, d\xi$$

$$B_{43} = k_4 h\int_{-1}^{1} \{[C_{13}h^2(\xi^2 - 1/5)/L_1 R_1 + C_{23}h^2(\xi^2 - 1/5)/L_2 R_2 + C_{33}s_8]s_4 L_1 + C_{16}h^2(\xi^2 - 1/5)/L_1 R_1 + C_{26}h^2(\xi^2 - 1/5)/L_2 R_2 + C_{36}s_8\} L_2 \, d\xi$$

$$B_{44} = h\int_{-1}^{1} [C_{36}s_9 + C_{66}h\xi/L_1 + (C_{33}s_9 + C_{36}h\xi/L_1)s_4 L_1]L_2 \, d\xi$$

$$B_{45} = h\int_{-1}^{1} [C_{36}s_{10} + C_{66}h\xi/L_2 + (C_{33}s_{10} + C_{36}h\xi/L_2)s_4 L_1]L_2 \, d\xi$$

$$B_{51} = h\int_{-1}^{1} [(C_{13}h\xi/L_1 + C_{33}s_6)s_5L_2 + C_{16}h\xi/L_1 + C_{36}s_6]L_1\,d\xi$$

$$B_{52} = h\int_{-1}^{1} [(C_{23}h\xi/L_2 + C_{33}s_7)s_5L_2 + C_{26}h\xi/L_2 + C_{36}s_7]L_1\,d\xi$$

$$B_{53} = k_4h\int_{-1}^{1} \{[C_{13}h^2(\xi^2-1/5)/L_1R_1 + C_{23}h^2(\xi^2-1/5)/L_2R_2 + C_{33}s_8]s_5L_2 + C_{16}h^2(\xi^2-1/5)/L_1R_1$$
$$+ C_{26}h^2(\xi^2-1/5)/L_2R_2 + C_{36}s_8\}L_1\,d\xi$$

$$B_{54} = h\int_{-1}^{1} [C_{36}s_9 + C_{66}h\xi/L_1 + (C_{33}s_9 + C_{36}h\xi/L_1)s_5L_2]L_1\,d\xi$$

$$B_{55} = h\int_{-1}^{1} [C_{36}s_{10} + C_{66}h\xi/L_2 + (C_{33}s_{10} + C_{36}h\xi/L_2)s_5L_2]L_1\,d\xi$$

$$D_{11} = h\int_{-1}^{1} [(C_{11}h\xi/L_1 + C_{13}s_6)h\xi + (C_{13}h\xi/L_1 + C_{33}s_6)s_6L_1]L_2\,d\xi$$

$$D_{12} = h\int_{-1}^{1} [(C_{12}h\xi/L_2 + C_{13}s_7)h\xi + (C_{23}h\xi/L_2 + C_{33}s_7)s_6L_1]L_2\,d\xi$$

$$D_{13} = k_4h\int_{-1}^{1} \{[C_{11}h^2(\xi^2-1/5)/L_1R_1 + C_{12}h^2(\xi^2-1/5)/L_2R_2 + C_{13}s_8]h\xi + [C_{13}h^2(\xi^2-1/5)/L_1R_1$$
$$+ C_{23}h^2(\xi^2-1/5)/L_2R_2 + C_{33}s_8]s_6L_1\}L_2\,d\xi$$

$$D_{14} = h\int_{-1}^{1} [(C_{13}s_9 + C_{16}h\xi/L_1)h\xi + (C_{33}s_9 + C_{36}h\xi/L_1)s_6L_1]L_2\,d\xi$$

$$D_{15} = h\int_{-1}^{1} [(C_{13}s_{10} + C_{16}h\xi/L_2)h\xi + (C_{33}s_{10} + C_{36}h\xi/L_2)s_6L_1]L_2\,d\xi$$

$$D_{22} = h\int_{-1}^{1} [(C_{22}h\xi/L_2 + C_{23}s_7)h\xi + (C_{23}h\xi/L_2 + C_{33}s_7)s_7L_2]L_1\,d\xi$$

$$D_{23} = k_4h\int_{-1}^{1} \{[C_{12}h^2(\xi^2-1/5)/L_1R_1 + C_{22}h^2(\xi^2-1/5)/L_2R_2 + C_{23}s_8]h\xi + [C_{13}h^2(\xi^2-1/5)/L_1R_1$$
$$+ C_{23}h^2(\xi^2-1/5)/L_2R_2 + C_{33}s_8]s_7L_2\}L_1\,d\xi$$

$$D_{24} = h\int_{-1}^{1} [(C_{23}s_9 + C_{26}h\xi/L_1)h\xi + (C_{33}s_9 + C_{36}h\xi/L_1)s_7L_2]L_1\,d\xi$$

$$D_{25} = h\int_{-1}^{1} [(C_{23}s_{10} + C_{26}h\xi/L_2)h\xi + (C_{33}s_{10} + C_{36}h\xi/L_2)s_7L_2]L_1\,d\xi$$

$$D_{33} = k_4^2h\int_{-1}^{1} \{[C_{11}h^2(\xi^2-1/5)/L_1R_1 + C_{12}h^2(\xi^2-1/5)/L_2R_2 + C_{13}s_8]h^2(\xi^2-1/5)/R_1$$
$$+ [C_{12}h^2(\xi^2-1/5)/L_1R_1 + C_{22}h^2(\xi^2-1/5)/L_2R_2 + C_{23}s_8]h^2\xi L_1/L_2R_2$$
$$+ [C_{13}h^2(\xi^2-1/5)/L_1R_1 + C_{23}h^2(\xi^2-1/5)/L_2R_2 + C_{33}s_8]s_8L_1\}L_2\,d\xi$$

$$D_{34} = k_4h\int_{-1}^{1} [(C_{13}s_9 + C_{16}h\xi/L_1)h^2(\xi^2-1/5)/R_1 + (C_{23}s_9 + C_{26}h\xi/L_1)h^2\xi L_1/L_2R_2$$
$$+ (C_{33}s_9 + C_{36}h\xi/L_1)s_8L_1]L_2\,d\xi$$

$$D_{35} = k_4h\int_{-1}^{1} [(C_{13}s_{10} + C_{16}h\xi/L_2)h^2(\xi^2-1/5)/R_1 + (C_{23}s_{10} + C_{26}h\xi/L_2)h^2\xi L_1/L_2R_2$$
$$+ (C_{33}s_{10} + C_{36}h\xi/L_2)s_8L_1]L_2\,d\xi$$

$$D_{44} = h\int_{-1}^{1} [(C_{36}s_9 + C_{66}h\xi/L_1)h\xi + (C_{33}s_9 + C_{36}h\xi/L_1)s_9L_1]L_2\,d\xi$$

$$D_{45} = h\int_{-1}^{1} [(C_{36}s_{10}+C_{66}h\xi/L_2)h\xi+(C_{33}s_{10}+C_{36}h\xi/L_2)s_9L_1]L_2\,d\xi$$

$$D_{55} = h\int_{-1}^{1} [(C_{36}s_{10}+C_{66}h\xi/L_2)h\xi+(C_{33}s_{10}+C_{36}h\xi/L_2)s_{10}L_2]L_1\,d\xi$$

$$G_{44} = k_2^2h(5/4)^2\int_{-1}^{1} (1-\xi^2)^2C_{44}L_1/L_2\,d\xi$$

$$G_{54} = k_1k_2h(5/4)^2\int_{-1}^{1} (1-\xi^2)^2C_{45}\,d\xi$$

$$G_{55} = k_1^2h(5/4)^2\int_{-1}^{1} (1-\xi^2)^2C_{55}L_2/L_1\,d\xi$$

Note the symmetry of the components: $A_{ij} = A_{ji}$, $D_{ij} = D_{ji}$ and $G_{ij} = G_{ji}$.

The transverse correction factors, as determined in section 3, are $k_1 = k_2 = \pi/\sqrt{10}, k_3 = \pi/\sqrt{12}$, and $k_4 = \pi/\sqrt{17/252}$.

APPENDIX E: COEFFICIENTS OF STIFFNESS AND MASS MATRICES $\mathbf{K}_{cyl}$ AND $\mathbf{M}_{cyl}$

The stiffness matrix coefficients, eqn (38), are given by

$$K_{11} = A_{11}\alpha^2+A_{55}n^2/a^2$$
$$K_{12} = (A_{12}+A_{54})\alpha n/a$$
$$K_{13} = A_{12}\alpha/a$$
$$K_{14} = (B_{12}+B_{54})\alpha n/a$$
$$K_{15} = B_{11}\alpha^2+B_{55}n^2/a^2$$
$$K_{16} = A_{13}\alpha/h$$
$$K_{17} = B_{13}\alpha/h^2$$
$$K_{22} = A_{44}\alpha^2+(A_{22}n^2+G_{44})/a^2$$
$$K_{23} = (A_{22}+G_{44})n/a^2$$
$$K_{24} = B_{44}\alpha^2+B_{22}n^2/a^2-G_{44}/a$$
$$K_{25} = (B_{45}+B_{21})\alpha n/a$$
$$K_{26} = A_{23}n/ha$$
$$K_{27} = B_{23}n/h^2a$$
$$K_{33} = G_{55}\alpha^2+G_{44}n^2/a^2+A_{22}/a^2$$
$$K_{34} = (B_{22}/a-G_{44})n$$
$$K_{35} = (B_{21}/a-G_{55})\alpha$$
$$K_{36} = A_{23}/ha$$
$$K_{37} = B_{23}/h^2a$$
$$K_{44} = D_{44}\alpha^2+D_{22}n^2/a^2+G_{44}$$
$$K_{45} = (D_{45}+D_{21})\alpha n/a$$
$$K_{46} = B_{32}n/ha$$
$$K_{47} = D_{23}n/h^2a$$
$$K_{55} = D_{11}\alpha^2+D_{55}n^2/a^2+G_{55}$$
$$K_{56} = B_{31}\alpha/h$$
$$K_{57} = D_{13}\alpha/h^2$$

$$K_{66} = A_{33}/h^2$$

$$K_{67} = B_{33}/h^3$$

$$K_{77} = D_{33}h^4.$$

The non-vanishing mass matrix coefficients, eqn (38), are given by

$$M_{11} = m_0$$
$$M_{15} = hm_1$$
$$M_{22} = m_0$$
$$M_{24} = M_{15}$$
$$M_{33} = m_0$$
$$M_{36} = m_1$$
$$M_{37} = -m_0/5 + m_2$$
$$M_{44} = h^2 m_2$$
$$M_{55} = M_{44}$$
$$M_{66} = m_2$$
$$M_{67} = -m_1/5 + m_3$$
$$M_{77} = m_0/25 - 2m_2/5 + m_4,$$

where the inertial coefficients, $m_n (n = 0, 1, \ldots, 4)$, are defined as

$$m_n = \rho h\{[1-(-1)^{n+1}]/(n+1) + [1-(-1)^{n+2}](1/R_1 + 1/R_2)h/(n+2) + [1-(-1)^{n+3}]h^2/R_1 R_2(n+3)\}.$$

Note: $K_{ij} = K_{ji}$ and $M_{ij} = M_{ji}$.